

Quantum Lower Bound for the Collision Problem with Small Range

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Abstract: We extend Aaronson and Shi’s quantum lower bound for the r -to-one collision problem. An r -to-one function is one where every element of the image has exactly r preimages. The r -to-one collision problem is to distinguish between one-to-one functions and r -to-one functions over an n -element domain.

Recently, Aaronson and Shi proved a lower bound of $\Omega((n/r)^{1/3})$ quantum queries for the r -to-one collision problem. Their bound is tight, but their proof applies only when the range has size at least $3n/2$. We give a modified version of their argument that removes this restriction.

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1 Introduction

How many quantum queries does it take to find a collision? A *collision* in a function is a pair of inputs that map to the same value. We consider the problem of finding a collision in an r -to-one function; i.e., a function where every element of the image has exactly r preimages. (We require that r be a divisor of n , the size of the input space.) The difficulty of this problem for a quantum computer has attracted much interest [1, 2, 4, 3, 6, 10].

In some cases, explicit information about a function may make it easier to find collisions. For example, if we know a function is periodic, we can find a collision using Shor’s algorithm [11]. Rather

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than use such explicit information, we focus on a *black-box* model: our only access to the function is as a quantum oracle. Brassard, Høyer, and Tapp [6] use Grover’s search [7] to find a collision in an r -to-one function in $O((n/r)^{1/3})$ quantum queries, an improvement over the $\Theta((n/r)^{1/2})$ classical queries needed. In this note, we are concerned with the matching lower bound.

For a lower bound, it is easier to consider a decision problem: the input function is guaranteed to be either one-to-one or r -to-one, and our task is to distinguish between these two cases. Aaronson [1] proved the first significant lower bound: $\Omega((n/r)^{1/5})$ queries.

More recently, Shi [10] proved a lower bound of $\Omega((n/r)^{1/3})$, given the additional condition that the size of the range of the function is at least $3n/2$. (In the case where the range is only n , Shi provides a lower bound of $\Omega((n/r)^{1/4})$). The proof is a novel application of the methods of Nisan and Szegedy [8] and Paturi [9] to the case where one cannot fully symmetrize the multivariate polynomials.

Our main result is a new version of this theorem, but without the additional constraint on the size of the range:

Theorem 1.1. *Let $n > 0$ and $r \geq 2$ be integers with $r \mid n$, and let a function from $[n]$ to $[n]$ be given as an oracle with the promise that it is either one-to-one or r -to-one. Then any quantum algorithm for distinguishing these two cases must evaluate the function $\Omega((n/r)^{1/3})$ times.*

The argument is very similar to that of Aaronson and Shi. (See [2] for a combined version of [1] and [10].) As stated above, we remove the requirement that the range be at least $3n/2$. Our proof is conceptually simpler for other reasons:

1. The natural automorphism group on the set of functions from $[n]$ to $[N]$ is $S_n \times S_N$. Our argument symmetrizes with respect to the entire group.
2. For technical reasons, Shi introduces an additional decision problem called Half- r -to-one, where one must distinguish between r -to-one functions and functions that are r -to-one on half the domain and one-to-one on the other half. We avoid using this Half- r -to-one problem.

An independent approach

Independent of this work, Ambainis [4] gave an alternate proof of [Theorem 1.1](#). His approach is more general: he shows that, given any lower bound for a symmetric function property with a restriction on the size of the range, we can remove that restriction.

Ambainis’s work, together with Shi’s paper, implies [Theorem 1.1](#). It is worth noting another consequence of those two papers: Aaronson and Shi prove that, given a black-box function f on n inputs whose range has size $\Omega(n^2)$, it takes $\Omega(n^{2/3})$ queries to determine if f is one-to-one. [Theorem 1.1](#) implies a similar result; the constant hidden in the $\Omega(n^2)$ term improves, but the dependence on n does not. Neither Aaronson and Shi [2] nor this paper gives a lower bound for element distinctness with small range.

However, Ambainis’s work gives a lower bound of $\Omega(n^{2/3})$ without any range restriction. Ambainis has also given a matching upper bound [3].

2 Preliminaries

2.1 Functions as quantum oracles

Let $n, N > 0$ be integers. Let $\mathcal{F}(n, N)$ be the set of functions from $[n]$ to $[N]$.

A function is given to us as a quantum oracle. We can perform a transformation O_f , which applies f to the contents of some of the quantum state:

$$O_f |i, j, z\rangle = |i, f(i) + j \pmod{N}, z\rangle .$$

Here z is a placeholder for the unaffected portion of the quantum state.

The query complexity of a quantum algorithm is the number of times it calls O_f . We think of our algorithm as alternating between $T + 1$ unitary operators and T applications of O_f .

Let $\delta_{i,j}(f)$ be 1 when $f(i) = j$ and 0 otherwise. Then, after T queries, the amplitude of each quantum base state is a degree- T polynomial in these $\delta_{i,j}(f)$. Hence, the acceptance probability $P(f)$ is a polynomial over $\delta_{i,j}$ of degree at most $2T$. The connection between quantum complexity and polynomial degree is due to Beals, et al. [5]; the application to functions using variables $\delta_{i,j}$ is due to Aaronson [1].

Note that this polynomial $P(f)$ is constrained to be in the interval $[0, 1]$ whenever the $\delta_{i,j}$ correspond to a valid input; i.e.,

$$\begin{aligned} \forall i, j, \quad & \delta_{i,j} \in \{0, 1\} , \\ \forall i, \quad & \sum_j \delta_{i,j} = 1 . \end{aligned} \tag{2.1}$$

The connection between polynomial degree and query complexity was first made by Nisan and Szegedy [8]. In their applications, they symmetrize over all permutations of the variables, reducing the multivariate polynomial to a univariate polynomial. They then apply results from approximation theory to prove a lower bound on the degree of the polynomial. Beals, et al. [5] follow the same approach.

A nice, general version of the approximation theory results was shown by Paturi [9]. Following Shi [10], we use a slight modification of Paturi's theorem:

Theorem 2.1 (Paturi). *Let $q(\alpha) \in \mathbb{R}[\alpha]$ be a polynomial of degree d . Let a and b be integers, $a < b$, and let $\xi \in [a, b]$ be a real number. If*

1. $|q(i)| \leq c_1$ for all integers $i \in [a, b]$, and
2. $|q(\lceil \xi \rceil) - q(\xi)| \geq c_2$ for some constant $c_2 > 0$,

then

$$d = \Omega(\sqrt{(\xi - a + 1)(b - \xi + 1)}) ,$$

where the hidden constant depends on c_1 and c_2 .

Note that, if the conditions of the theorem are met for any ξ , we have $d = \Omega(\sqrt{b - a})$. If they are met for some $\xi \approx (a + b)/2$, then $d = \Omega(b - a)$.

In our setting, the automorphism group for the variables $\delta_{i,j}$ is $S_n \times S_N$. If we symmetrize with respect to this group, we do not immediately obtain a univariate polynomial. Hence, we will have to work harder to apply [Theorem 2.1](#).

For $\sigma \in S_n$, $\tau \in S_N$, we define $\Gamma_\tau^\sigma : \mathcal{F}(n, N) \rightarrow \mathcal{F}(n, N)$ by

$$\Gamma_\tau^\sigma(f) = \tau \circ f \circ \sigma .$$

Let $P(f)$ be an acceptance polynomial as above. We can write P as a sum $\sum_S C_S I_S(f)$, where S ranges over subsets of $[n] \times [N]$ and

$$I_S = \prod_{(i,j) \in S} \delta_{i,j} .$$

By [\(2.1\)](#), we may assume that each pair $(i, j) \in S$ has a distinct value of i ; we thus write

$$I_S = \prod_{k=1}^t \prod_{i \in S_k} \delta_{i,j_k} , \tag{2.2}$$

where the sets S_k are disjoint. The degree of the monomial is $\sum_k |S_k|$.

2.2 Some special functions

We now define a collection of functions which are a -to-one on part of the domain, and b -to-one on the rest of the domain. (These will enable us to interpolate between one-to-one and r -to-one functions.)

Fix $N \geq n > 0$. We say that a triple (m, a, b) of integers is *valid* if $0 \leq m \leq n$, $a \mid m$, and $b \mid (n - m)$. For any such valid triple, we have a function $f_{m,a,b} \in \mathcal{F}(n, N)$, given by

$$f_{m,a,b} = \begin{cases} \lceil i/a \rceil & 1 \leq i \leq m , \\ N - \lfloor (n - i)/b \rfloor & m < i \leq n . \end{cases}$$

So $f_{m,a,b}$ is a -to-one on m points, and b -to-one on the remaining $n - m$ points. (Since $N \geq n$, the two parts of the range do not overlap.)

Note that our $f_{m,a,b}$ plays the same role as Aaronson and Shi's $f_{m,g}$, with $a = g$ and $b = 2$.

We now examine the behavior of $f_{m,a,b}$ after we symmetrize by all of $S_n \times S_N$.

Lemma 2.2. *Let $P(f)$ be a degree- d polynomial in $\delta_{i,j}$. For a valid triple (m, a, b) , define $Q(m, a, b)$ by*

$$Q(m, a, b) = \mathbf{E}_{\sigma, \tau} [P(\Gamma_\tau^\sigma(f_{m,a,b}))] .$$

Then Q is a degree- d polynomial in m, a, b .

The key new step in this paper lies in the proof of [Lemma 2.2](#). To show that the expected value $Q(m, a, b)$ is a polynomial, we break down S_N into a union of disjoint events A_U . We then write $Q(m, a, b)$ as a sum over all U , and we show that each term in the sum is a polynomial in m, a , and b .

Definition 2.3. For integers k, ℓ , let $\ell^{\underline{k}}$ denote the falling power $\ell(\ell - 1) \cdots (\ell - k + 1)$.

Proof of Lemma 2.2. It suffices to prove the lemma in the case where P is a monomial I_S . We write I_S in the form (2.2); then $d = |S|$. We write $s_k = |S_k|$.

For each subset $U \subseteq [t]$, let A_U be the following event: for each $k \in U$, $\tau^{-1}(j_k) \leq m/a$; for each $k \notin U$, $\tau^{-1}(j_k) \geq N - (n-m)/b + 1$.

Clearly the events A_U are disjoint. If $I_S(\Gamma_\tau^\sigma(f_{m,a,b}))$ is nonzero, then every $\tau^{-1}(j_k)$ must lie in the range of $f_{m,a,b}$, so some event A_U must occur. Hence, we write

$$Q(m, a, b) = \sum_{U \subseteq [t]} \Pr(A_U) Q_U(m, a, b) ,$$

where

$$Q_U(m, a, b) = \mathbf{E}_{\sigma, \tau} [I_S(\Gamma_\tau^\sigma(f_{m,a,b})) \mid A_U] .$$

Choose some U , and let $u = |U|$. Then $\Pr(A_U)$ is given by

$$\Pr(A_U) = \frac{\binom{m}{a}^u \binom{n-m}{b}^{t-u}}{N^t} ,$$

which is a rational function in m, a, b . The numerator has degree t , and the denominator is $a^u b^{t-u}$.

Also,

$$Q_U(m, a, b) = \frac{1}{n^d} \prod_{k \in U} a^{s_k} \prod_{k \notin U} b^{s_k} .$$

This is a polynomial in a, b of degree d ; furthermore Q_U is divisible by $a^u b^{t-u}$.

Hence, for each U , $\Pr(A_U)Q_U$ is a degree- d polynomial in m, a, b . Therefore $Q(m, a, b)$ is itself a degree- d polynomial. This concludes the lemma. \square

3 Main Proof

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let \mathcal{A} be an algorithm which distinguishes one-to-one from r -to-one in T queries, and let $P(f)$ be the corresponding acceptance probability. $P(f)$ is a polynomial in $\delta_{i,j}$ of degree at most $2T$. Let $Q(m, a, b)$ be formed from P as in Lemma 2.2, and let $d = \deg Q$; we have $d \leq 2T$.

For any σ, τ , we know that $\Gamma_\tau^\sigma(f_{m,a,b})$ is a valid function. If $a = b$, this function is a -to-one. We conclude the following:

1. $0 \leq Q(m, a, b) \leq 1$ whenever (m, a, b) is a valid triple.
2. $0 \leq Q(m, 1, 1) \leq 1/3$ for any m .
3. $2/3 \leq Q(m, r, r) \leq 1$ for any m such that $r \mid m$.

The remainder of the proof consists of proving that $\deg Q = \Omega((n/r)^{1/3})$. We will take $M \approx m/2$, and we will examine either the univariate polynomial $Q(M, 1, rx)$ or $Q(M, rx, r)$ (depending on the value of $Q(M, 1, r)$). If this polynomial remains bounded for large values of x , we can apply [Theorem 2.1](#). Otherwise, we can use [Theorem 2.1](#) on the first argument to Q . Either way, we get a lower bound on d .

For simplicity of exposition, we begin with the case $r = 2$. Let $M = 2\lfloor n/4 \rfloor$. We ask: is $Q(M, 1, 2) \geq 1/2$? In other words: does our algorithm accept (at least half the time) an input which is one-to-one on half the domain, and two-to-one on the other half?

Case I: $Q(M, 1, 2) \geq 1/2$. Let $g(x) = Q(M, 1, 2x)$, and let k be the least positive integer for which $|g(k)| \geq 2$. Then we have $g(x)$ between -2 and 2 for all positive integers $x < k$, and $g(1) - g(1/2) \geq 1/6$ by assumption. Let $c = 2k$. By [Theorem 2.1](#), we have

$$d = \Omega(\sqrt{k}) = \Omega(\sqrt{c}) . \quad (3.1)$$

Now, we consider the polynomial $h(i) = Q(n - ci, 1, c)$. For any integer i in the range $0 \leq i \leq \lfloor n/c \rfloor$, the triple $(n - ci, 1, c)$ is valid, so $0 \leq h(i) \leq 1$. But

$$\left| h\left(\frac{n-M}{c}\right) \right| = |Q(M, 1, c)| = |g(k)| \geq 2 .$$

We conclude, by [Theorem 2.1](#), that

$$d = \Omega(n/c) . \quad (3.2)$$

Case II: $Q(M, 1, 2) < 1/2$. Now, let $g(x) = Q(M, 2x, 2)$. Let k be the least positive integer for which $|g(k)| \geq 2$, and let $c = 2k$. We have $g(1) - g(1/2) \geq 1/6$; as in Case I, we obtain (3.1) using [Theorem 2.1](#).

Next, we consider $h(i) = Q(ci, c, 2)$. For any integer i in the range $0 \leq i \leq \lfloor n/c \rfloor$, the triple $(ci, c, 2)$ is valid (both n and c are even), so $0 \leq h(i) \leq 1$. But $|h(M/c)| = |g(k)| \geq 2$. Again, as in Case I, we obtain (3.2) using [Theorem 2.1](#).

In either case, we use (3.1) and (3.2) to obtain $d = \Omega(n^{1/3})$. We could divide into cases (depending on whether $c \geq n^{2/3}$), or we could simply square (3.1) and multiply by (3.2) to obtain $d^3 = \Omega(n)$.

For general r , the setup is almost identical: we let $M = r\lfloor \frac{n}{2r} \rfloor$ and split into cases based on whether $Q(M, 1, r) \geq 1/2$.

Case I: $Q(M, 1, r) \geq 1/2$. Let $g(x) = Q(M, 1, rx)$, let k be the least positive integer for which $|g(k)| \geq 2$, and let $c = rk$. We have $g(1) - g(1/r) \geq 1/6$, so [Theorem 2.1](#) yields

$$d = \Omega(\sqrt{k}) = \Omega(\sqrt{c/r}) . \quad (3.3)$$

Next, we let $h(i) = Q(n - ci, 1, c)$. As in the $r = 2$ analysis above, we conclude (3.2).

Case II: $Q(M, 1, r) < 1/2$. Now, let $g(x) = Q(M, rx, r)$, let k be the least integer for which $|g(k)| \geq 2$, and let $c = rk$. We have $g(1) - g(1/r) \geq 1/6$; as in Case I, we obtain (3.3) using [Theorem 2.1](#).

Next, we take $h(i) = Q(ci, c, r)$. For any integer i in the range $0 \leq i \leq \lfloor n/c \rfloor$, the triple (ci, c, r) is valid; note that $n - ci$ must be a multiple of r . But $|h(M/c)| = |g(k)| \geq 2$. So, as in the $r = 2$ analysis, we get (3.2).

In either case, we square (3.3) and multiply by (3.2) to obtain $d^3 = \Omega(n/r)$ as desired. \square

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References

- [1] * SCOTT AARONSON: Quantum lower bound for the collision problem. In *Proc. of the 34th ACM STOC*, pp. 635–642, 2002. [[STOC:509907.509999](#), [arXiv:quant-ph/0111102](#)]. 1, 1, 2.1, 2
- [2] * SCOTT AARONSON AND YAORYUN SHI: Quantum lower bounds for the collision and the element distinctness problems. *Journal of the ACM*, 51(4):595–605, 2004. Based on [10] and [1]. [[JACM:1008735](#)]. 1, 1, 1
- [3] * ANDRIS AMBAINIS: Quantum walk algorithm for element distinctness. In *Proc. of the 45th IEEE FOCS*, pp. 22–31, 2004. [[FOCS:10.1109/FOCS.2004.54](#), [arXiv:quant-ph/0311001](#)]. 1, 1
- [4] * ANDRIS AMBAINIS: Quantum lower bounds for collision and element distinctness with small range. *Theory of Computing*, 1(3), 2005. To appear. [[ToC:v001/a003](#), [arXiv:quant-ph/0305179](#)]. 1, 1
- [5] * BOB BEALS, HARRY BUHRMAN, RICHARD CLEVE, MICHELE MOSCA, AND RONALD DE WOLF: Quantum lower bounds by polynomials. In *Proc. of the 39th IEEE FOCS*, pp. 352–361, 1998. [[FOCS:10.1109/SFCS.1998.743485](#), [arXiv:quant-ph/9802049](#)]. 2.1, 2.1
- [6] * GILLES BRASSARD, PETER HØYER, AND ALAIN TAPP: *Quantum Cryptanalysis of Hash and Claw-Free Functions*, volume 1380 of *Lecture Notes in CS*, pp. 163–169. Springer-Verlag, 1998. [[LATIN:11bhjthw46dxl2qa](#), [arXiv:quant-ph/9805082](#)]. 1
- [7] * LOV K. GROVER: A fast quantum mechanical algorithm for database search. In *Proc. of the 28th ACM STOC*, pp. 212–219, 1996. [[STOC:237814.237866](#), [arXiv:quant-ph/9605043](#)]. 1
- [8] * NOAM NISAN AND MÁRIÓ SZEGEDY: On the degree of boolean functions as real polynomials. *Computational Complexity*, 4:301–313, 1994. [[STOC:129757](#)]. 1, 2.1
- [9] * RAMAMOCHAN PATURI: On the degree of polynomials that approximate symmetric boolean functions. In *Proc. of the 24th ACM STOC*, pp. 468–474, 1992. [[STOC:129758](#)]. 1, 2.1
- [10] * YAORYUN SHI: Quantum lower bounds for the collision and the element distinctness problems. In *Proc. of the 43th IEEE FOCS*, pp. 513–519, 2002. [[FOCS:10.1109/SFCS.2002.1181975](#), [arXiv:quant-ph/0112086](#)]. 1, 1, 2.1, 2
- [11] * PETER W. SHOR: Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 26(5):1484–1509, 1997. [[SICOMP:29317](#)]. 1

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